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TWO FAMILIES OF PERIODIC SOLUTIONS IN THE ROZE PROBLEM

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The Routh method is used to prove the stability of a steady motion in the Roze problem [1, 2]. Liapunov theorem on holomorphic integral is used to construct the periodic solutions in the neighborhood of a stable unperturbed motion.

We consider the motion of a system consisting of two heavy Lagrange gyroscopes, about a fixed point lying on the axis of dynamic symmetry of the first gyroscope. We shall assume that the axes of dynamic symmetry of the tops are connected by a cylindrical hinge which keeps them in some vertical plane but allows them to assume any inclination with respect to the vertical. Let s denote the distance between the fixed point and the hinge, l_1 the distance between the fixed point and the center of mass of the first gyroscope and l_2 the distance between the fixed point and the center of mass of the second gyroscope. A_1', A_2 and C_1, C_2 are the equatorial and axial moments of inertia of the gyroscopes.

The above system has five degrees of freedom and its position in space can be defined in terms of the corresponding angles of nutation θ_1, θ_2, $0 \leqslant \theta_2 \leqslant 2\pi$, angles of self-rotation φ_1 and φ_2, and the angle of general precession ψ. The Lagrangian function has the form

$$L = \frac{1}{2}\sum_{i=1}^{2}[A_i(\theta_i^{\cdot 2} + \psi^{\cdot 2}\sin^2\theta_i) + C_i(\varphi_i^{\cdot} + \psi^{\cdot}\cos\theta_i)^2] + \tag{1}$$

$$B[\theta_1^{\cdot}\theta_2^{\cdot}\cos(\theta_1 - \theta_2) + \psi^{\cdot 2}\sin\theta_1\sin\theta_2] -$$
$$g[(m_1 l_1 + m_2 s)\cos\theta_1 + m_2 l_2\cos\theta_2]$$
$$A_1 = A_1' + m_2 s^2, \quad B = m_2 l_2 s$$

where m_1 and m_2 denote the masses of the gyroscopes and g is the acceleration due to gravity.

The ignorable coordinates φ_1, φ_2 and ψ have the corresponding first integrals (n_1, n_2 and n are integration constants)

$$C_i(\varphi_i^{\cdot} + \psi^{\cdot}\cos\theta_i) = n_i, \quad i = 1, 2, \quad N\psi^{\cdot} = \nu$$
$$N = A_1\sin^2\theta_1 + A_2\sin^2\theta_2 + 2B\sin\theta_1\sin\theta_2, \quad \nu = n - n_1\cos\theta_1 - n_2\cos\theta_2$$

Eliminating from (1) the ignorable coordinates, we obtain the Routh function

$$R = L - n_1\varphi_1^{\cdot} - n_2\varphi_2^{\cdot} - n\psi^{\cdot} = {}^1/_2[A_1\theta_1^{\cdot 2} + A_2\theta_2^{\cdot 2} + 2B\theta_1^{\cdot}\theta_2^{\cdot}\cos(\theta_1 - \theta_2)] - W(\theta_1, \theta_2, n_1, n_2, n)$$

$$W = \frac{1}{2}\frac{\nu^2}{N} + g[(m_1 l_1 + m_2 s)\cos\theta_1 + m_2 l_2\cos\theta_2]$$

The steady motions of the initial system are obtained from the conditions [1, 2]

$$\partial W/\partial\theta_i = 0, \quad i = 1, 2 \tag{2}$$

Let the initial conditions be chosen so that the relations (2) are satisfied. Then the system performs a steady motion

$$\theta_1 = \theta_{10}, \quad \theta_2 = \theta_{20}, \quad \varphi_1 = \varphi_{10} + \varphi_{10}^{\cdot}t, \quad \varphi_2 = \varphi_{20} + \varphi_{20}^{\cdot}t, \quad \psi = \psi_0 + \psi_0 t \tag{3}$$

We shall regard (3) as an unperturbed motion, and investigate its stability using the Routh method. Assuming in the perturbed motion $\theta_1 = \theta_{10} + x_1$, $\theta_2 = \theta_{20} + x_2$, $\theta_1^{\cdot} = x_1^{\cdot}$, $\theta_2^{\cdot} = x_2^{\cdot}$, we find

$$W(\theta_1, \theta_2) - W(\theta_{10}, \theta_{20}) = \frac{1}{2}\left[\left(\frac{\partial^2 W}{\partial\theta_1^2}\right)_0 x_1^2 + \left(\frac{\partial^2 W}{\partial\theta_2^2}\right)_0 x_2^2 + 2\left(\frac{\partial^2 W}{\partial\theta_1\partial\theta_2}\right)_0 x_1 x_2\right] + \ldots$$

The function $W(\theta_1, \theta_2)$ has a minimum $W(\theta_{10}, \theta_{20})$ provided that

$$\left(\frac{\partial^2 W}{\partial\theta_1^2}\right)_0 > 0, \quad \left(\frac{\partial^2 W}{\partial\theta_1^2}\right)_0 \left(\frac{\partial^2 W}{\partial\theta_2^2}\right)_0 > \left(\frac{\partial^2 W}{\partial\theta_1\partial\theta_2}\right)_0 \tag{4}$$

We note that the set of solutions of the inequalities (4) is nonempty. For example, all $0 < \theta_{i0} < \pi$ $(i = 1, 2)$ satisfy (4). However, we shall not analyze these inequalities in detail. Using the Routh theory on stability of steady motions and its Liapunov complement [3] we conclude, that the steady motions are stable in $\theta_1, \theta_2, \theta_1^{\cdot}, \theta_2^{\cdot}, \varphi_1^{\cdot}, \varphi_2^{\cdot}, \psi^{\cdot}$ provided that the inequalities (4) hold.

The equations of perturbed motion have the form

$$A_1 x_1^{\cdot\cdot} + B x_2^{\cdot\cdot} = -\partial W / \partial x_1, \quad A_2 x_2^{\cdot\cdot} + B x_1^{\cdot\cdot} = -\partial W / \partial x_2 \tag{5}$$

Since the right-hand sides of (5) are functions holomorphic in x_1 and x_2, we have

$$x_i^{\cdot\cdot} = p_{i1}x_1 + p_{i2}x_2 + q_{i1}x_1^2 + q_{i2}x_1x_2 + q_{i3}x_2^2 + \ldots, \quad i = 1, 2 \tag{6}$$

$$p_{1i} = \frac{B\gamma_{i2} - A_2\gamma_{1i}}{D}, \quad p_{2i} = \frac{B\gamma_{1i} - A_1\gamma_{i2}}{D}, \quad i = 1, 2$$

$$q_{11} = \frac{Bq_{12}' - A_2 q_{11}'}{2D}, \quad q_{12} = \frac{Bq_{21}' - A_2 q_{12}'}{D}, \quad q_{13} = \frac{Bq_{22}' - A_2 q_{21}'}{2D}$$

$$q_{21} \frac{Bq_{11}' - A_1 q_{12}'}{2D}, \quad q_{22} = \frac{Bq_{12}' - A_1 q_{21}'}{D}, \quad q_{23} = \frac{Bq_{21}' - A_1 q_{22}'}{2D}$$

$$\gamma_{11} = \left(\frac{\partial^2 W}{\partial x_1^2}\right)_0, \quad \gamma_{22} = \left(\frac{\partial^2 W}{\partial x_2^2}\right)_0, \quad \gamma_{12} = \gamma_{21} = \left(\frac{\partial^2 W}{\partial x_1 \partial x_2}\right)_0$$

$$q_{11}' = \left(\frac{\partial^3 W}{\partial x_1^3}\right)_0, \quad q_{22}' = \left(\frac{\partial^3 W}{\partial x_2^3}\right)_0, \quad q_{12}' = \left(\frac{\partial^3 W}{\partial x_1^2 \partial x_2}\right)_0$$

$$q_{21}' = (\partial^3 W / \partial x_2^2 \partial x_1)_0, \quad D = A_1 A_2 - B^2$$

The energy integral

$$E - E_0 = {}^1/_2(A_1 x_1^{\cdot 2} + A_2 x_2^{\cdot 2} + 2B x_1^{\cdot} x_2^{\cdot} + \gamma_{11}x_1^2 + \gamma_{22}x_2^2 + 2\gamma_{12}x_1x_2) + \ldots$$

is sign definite with respect to the variables in question, provided that $\gamma_{11} > 0$, $\gamma_{11}\gamma_{22} - \gamma_{12}^2 > 0$. The defining equation of the system (6) has the form

$$\Delta(\lambda) = \lambda^4 (A_1 A_2 - B^2) + \lambda^2 (A_1\gamma_{22} + A_2\gamma_{11} - 2B\gamma_{12}) + (\gamma_{11}\gamma_{22} - \gamma_{12}^2) = 0 \tag{7}$$

Let Eq. (7) have two pairs of pure imaginary roots $\pm \lambda_i \sqrt{-1}$. This happens, e. g. when $0 < \theta_{i0} < \pi$ $(i = 1, 2)$. In this case if one of the ratios λ_1/λ_2, λ_2/λ_1 is not an integer, the Liapunov theorem on holomorphic integral is used and each pair of the pure imaginary roots $\pm \lambda_i \sqrt{-1}$ $(i = 1, 2)$ has a corresponding single periodic solution of the system (5), (6) with two arbitrary constants [4]. Each periodic solution can be written in terms of periodic series in powers of some arbitrary constant c_i. We shall obtain the first two terms of these series. Let us set

$$T_i = 2\pi/\lambda_i\,[1 + h_{2i}c_i^2 + h_{3i}c_i^3 + \ldots]$$
$$\tau = 2\pi\,/\,T_i\,(t - t_{0i}), \quad i = 1, 2$$

Having transformed Eqs. (6) to the form

$$\frac{d^2x_s}{d\tau^2} = \frac{1}{\lambda_i^2}\,[1 + h_{2i}(c_i)^2 + \ldots]\,\{p_{s1}x_1 + p_{s2}x_2 + \overline{X}_s\}$$

we seek their solutions in the form of series

$$x_s = x_{si}^{(1)}c_i + x_{si}^{(2)}(c_i^2) + \ldots, \quad s = 1, 2$$

The coefficients $x_{1i}^{(1)}$ and $x_{2i}^{(1)}$ are found from the homogeneous linear equations (a_{1i} and b_{1i} are arbitrary constants)

$$a_{2i} = -(p_{11} + \lambda_i^2)\,a_{1i}\,/\,p_{12}, \quad b_{2i} = -(p_{11} + \lambda_i^2)\,b_{1i}\,/\,P_{12}$$
$$x_{1i}^{(1)} = a_{1i}\cos\tau + b_{1i}\sin\tau, \quad x_{2i}^{(1)} = a_{2i}\cos\tau + b_{2i}\sin\tau$$

Equations for the functions $x_{1i}^{(2)}$ and $x_{2i}^{(2)}$ can be written as follows:

$$\frac{d^2x_s^2}{d\tau^2} = \frac{1}{\lambda_s^2}\,[(p_{s1}x_1^{(2)} + p_{s2}x_2^{(2)}) + (A_{s2} + A_{s2}\cos 2\tau + B_{s2}\sin 2\tau)]$$

$$A_{s2} = \delta_{si}\,(a_{1i}^2 - b_{1i}^2), \quad B_{s2} = 2\delta_{si}a_{1i}b_{1i}$$

$$\delta_{si} = \frac{1}{2p_{12}^2}\,[q_{s1}p_{12}^2 + q_{s3}(p_{11} + \lambda_i^2)^2 - q_{s2}p_{s2}(p_{11} + \lambda_i^2)]$$

therefore $x_{si}^{(2)}$ is found in the form

$$x_{si}^{(2)} = a_{s2}^{(0)} + a_{s2}^{(2)}\cos 2\tau + b_{s2}^{(2)}\sin 2\tau, \quad s = 1, 2; \quad i = 1, 2$$

The coefficients $a_{s2}^{(0)}$, $a_{s2}^{(2)}$, $b_{s2}^{(2)}$ are

$$a_{12}^{(0)} = -[A_{12}p_{22} - A_{22}p_{12}]\,/\,\Delta_1, \quad a_{12}^{(2)} = -[A_{12}\,(p_{22} + 4\lambda_i^2) - A_{22}p_{12}]\,/\,\Delta_2$$

$$b_{12}^{(2)} = [B_{12}(p_{22} + 4\lambda_i^2) - B_{22}p_{12}]\,/\,\Delta_2, \quad a_{22}^{(0)} = [A_{12}p_{12} - A_{22}p_{11}]\,/\,\Delta_1$$

$$a_{22}^{(2)} = [A_{12}p_{21} - A_{22}\,(p_{11} + 4\lambda_i^2)]\,/\,\Delta_2, \quad b_{22}^{(2)} = [B_{12}p_{21} - B_{22} \times (p_{11} + 4\lambda_i^2)]\,/\,\Delta_2$$

$$\Delta_1 = p_{11}p_{22} - p_{12}p_{21}, \quad \Delta_2 = (p_{11} + 4\lambda_i^2)(p_{22} + 4\lambda_i^2) - p_{12}p_{21}$$

Thus we have shown that when condition (2) holds, two families of periodic solutions may exist and, according to Liapunov, they are constructed in the form of trigonometric series.

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